

Linear constraints from generally covariant systems with quadratic constraints

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How to make compatible both boundary and gauge conditions for generally covariant theories using the gauge symmetry generated by first class constraints is studied. This approach employs *finite* gauge transformations in contrast with previous works which use infinitesimal ones. Two kinds of variational principles are taken into account; the first one features non-gauge-invariant actions whereas the second includes fully gauge-invariant actions. Furthermore, it is shown that it is possible to rewrite fully gauge-invariant actions featuring first class constraints quadratic in the momenta into first class constraints linear in the momenta (and homogeneous in some cases) due to the full gauge invariance of their actions. This shows that the gauge symmetry present in generally covariant theories having first class constraints quadratic in the momenta is not of a different kind with respect to the one of theories with first class constraints linear in the momenta if fully gauge-invariant actions are taken into account for the former theories. These ideas are implemented for the parametrized relativistic free particle, parametrized harmonic oscillator, and the $SL(2, R)$ model.

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I. INTRODUCTION

The compatibility of both gauge and boundary conditions for gauge theories is a key point that must be satisfied to have a well-defined dynamics. It could happen that the boundary conditions chosen for a certain action were incompatible with the choice made to fix the gauge freedom of the system. If this were the case, both gauge and boundary conditions could be made compatible using the infinitesimal gauge symmetry generated by the first class constraints [1]. It is important to recall that the gauge transformation associated with the first class constraints is not a sufficient condition to achieve that goal; an extra input is needed: the transversality condition, which precisely allows the use of the gauge transformation in the method. The infinitesimal procedure is completely systematic, being easily generalized to field theory [2]. However, is there any difference if finite gauge transformations are taken into account? On one hand, they are really important in a nonperturbative treatment of the full symmetry of gauge systems which has relevance both classically and quantum mechanically. On the other hand, the finite gauge transformations include the large gauge transformations; these transformations are not connected to the identity, therefore their effects are not observed by the infinitesimal procedure. So, for example, the application of the infinitesimal procedure to the $SL(2, R)$ model implies that its diffeomorphism constraint (linear and homogeneous in the momenta) does not contribute to the boundary term. However, the finite approach developed here shows that it really does contribute to it. Obviously, this contribution cannot be neglected if a complete analysis of the full gauge symmetry is required. In this way, the procedure here presented can be useful in the analysis of gauge-invariant systems with nontrivial topological spaces or in systems with singular boundary conditions. That is why *finite* gauge transformations are really important.

To implement in the action the finite gauge transformations, two kinds of variational principles are analyzed: the first one features non-gauge-invariant actions whereas the ones of the second are fully gauge-invariant.

The first type of variational principle based on non-gauge-invariant actions is conceptually the finite extension of the method of Ref. [1]. Like there, the transversality condition is not necessary in the case of noncanonical gauges but is needed if algebraic gauge conditions are taken into account. Even though “gauge-invariant” actions are used in Ref. [1], the meaning of gauge invariance adopted there is not the standard one. So, strictly speaking, variational principles with fully gauge-invariant actions were not considered there either in their infinitesimal or finite versions.

The second type of variational principle introduced here includes fully gauge-invariant actions, where gauge invariance has the usual meaning. Of course, the same issues concerning the transversality condition are those of Ref. [1]

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for the same reasons that apply in that case. This second type of variational principle is analyzed in both its finite and infinitesimal versions. Finally, these two kinds of variational principles are applied to the parametrized relativistic free particle, parametrized harmonic oscillator, and the $SL(2, R)$ model with two non-commuting Hamiltonian constraints introduced in Ref.[3]. This is the first result of the paper displayed in detail in Sec. II.

The second point analyzed here is deeper. Generally covariant theories usually have first class constraints quadratic in the momenta. Examples per excellence of these kinds of theories are general relativity and string theory as well as many toy models with finite degrees of freedom found in the literature. It should be expected that their Hamiltonian actions were fully gauge-invariant under the gauge symmetry. However, this is not so, rather, the gauge transformed actions of these theories are equal to the original ones plus boundary terms because of the Hamiltonian constraints quadratic in the momenta. Therefore one usually handles gauge theories coming from nonfully gauge-invariant actions which is an ugly situation from the point of view of gauge theories. In this paper gauge symmetry is taken seriously and variational principles with fully gauge-invariant actions are constructed by adding suitable boundary terms to the non-gauge-invariant actions. By introducing these boundary terms into the integral the quadratic constraints are modified in a nice way: they become linear in the momenta (and homogeneous in some cases) in the new canonical variables (Q^i, P_i) naturally induced by the boundary terms, which is a beautiful result just coming from the full gauge invariance of their actions. Thus first class constraints quadratic in the momenta are *not* the distinguishing mark of generally covariant theories if fully gauge-invariant actions are taken into account (cf. Ref. [4]). This result raises the question: what then is a Hamiltonian constraint? If generally covariant systems endowed with fully gauge-invariant actions and first class constraints quadratic in the momenta can be written in terms of first class constraints linear in the momenta like Yang-Mills theories, then how does one distinguish a genuine Yang-Mills-like theory from one coming from a “linearization” in the sense explained of a generally covariant system? Does it make sense to distinguish both types of theories just by the form that momenta enter in their constraints? Even though it is not possible to give a definitive answer to these questions from the present result, it is hoped that it can contribute to make clear the meaning of Hamiltonian constraints in generally covariant systems. In particular, the transformation of the constraints could be useful to find in some cases a new set of solutions of physical states through the Dirac condition $G_a(Q^i, P_i)|\psi\rangle = 0$. This constitutes the second result of the present paper displayed in detail in Sec. III. Finally, our conclusions are presented in Sec. IV.

Let us begin by recalling the canonical transformation induced when a boundary term is present, because this is the heart of the methods used in this paper. From now on, it is considered a generally covariant system determined by an action of the form [5]

$$S[q^i, p_i, \lambda^a] = \int_{\tau_1}^{\tau_2} d\tau (p_i \dot{q}^i - H_E) - B|_{\tau_1}^{\tau_2}, \quad i = 1, \dots, N, \quad (1)$$

where $H_E = H_0 + \lambda^a G_a$ is the extended Hamiltonian, G^a are first class constraints, and H_0 is the canonical first class Hamiltonian,

$$\begin{aligned} \{G_a, G_b\} &= C_{ab}{}^c G_c, \\ \{H_0, G_a\} &= V_a{}^b G_b, \quad a, b, c = 1, \dots, M. \end{aligned} \quad (2)$$

Therefore the system has $D = N - M$ continuous physical degrees of freedom ($2D$ in the reduced phase space). The boundary term $B(q, p, \tau)$ determines a complete set of commuting variables $Q^i(q, p, \tau)$ fixed at the end points τ_1 and τ_2 ,

$$\begin{aligned} Q^i[q(\tau_\alpha), p(\tau_\alpha), \tau_\alpha] &= Q_\alpha^i, \quad \alpha = 1, 2, \\ \{Q^i, Q^j\} &= 0, \quad (\text{at equal } \tau\text{'s}). \end{aligned} \quad (3)$$

These commuting variables are equal in number as the q 's (“completeness”) and could be, for instance, the q 's themselves ($Q^i = q^i$) or any other combination of q 's and p 's satisfying the commutation condition in the Poisson brackets sense. The relationship between these variables and the boundary term is given by

$$p_i \delta q^i - H_E \delta \tau - \delta B = P_i \delta Q^i - \mathcal{H}_E \delta \tau, \quad (4)$$

where the P 's are the momenta conjugate to the Q 's, and \mathcal{H}_E is the new extended Hamiltonian. Taking independent variations of q^i , p_i , and τ yields

$$\begin{aligned} p_i - \frac{\partial B}{\partial q^i} - P_j \frac{\partial Q^j}{\partial q^i} &= 0, \\ P_j \frac{\partial Q^j}{\partial p_i} + \frac{\partial B}{\partial p_i} &= 0, \\ \mathcal{H}_E - H_E - \frac{\partial B}{\partial \tau} - \frac{\partial Q^j}{\partial \tau} P_j &= 0, \end{aligned} \quad (5)$$

which establish the relationship between the new phase space variables (Q^i, P_i) with the initial ones (q^i, p_i) . There is no uniqueness in the solution of these equations, in a similar way to the fact that a canonical transformation can be obtained using different generating functions.

II. COMPATIBILITY BETWEEN BOUNDARY AND GAUGE CONDITIONS USING FINITE GAUGE TRANSFORMATIONS

1. Non-gauge-invariant actions. It was stated in the Introduction that this variational principle featuring finite gauge transformations is, essentially, the finite version of the approach of Ref. [1] based on non-gauge-invariant actions. From a technical perspective the only remark is that the interplay between first class constraints linear and homogeneous in the momenta and quadratic (or higher) in the momenta could imply a contribution to the boundary term of the former if finite gauge transformations are taken into account [see the example of the $SL(2, R)$ model in this section]. This is a key difference between the infinitesimal and finite versions because in the infinitesimal approach constraints linear and homogeneous in the momenta never contribute to boundary terms.

To be precise, this method is an analogue to the one in Ref. [1]. There, authors use an “hybrid” variational principle because the action and the boundary conditions are expressed in terms of the original set of variables, however, the right gauge condition is expressed in terms of the gauge related ones. Here, on the other hand, the standard procedure is followed, namely, the action, the boundary conditions, and the gauge condition are written in terms of the *same* set of canonical variables: the gauge related ones. Before going to the description of the method, boundary conditions (3) and algebraic gauge conditions $\chi_a(q^i, p_i) = 0$ are assumed not compatible to each other. Then, using the finite gauge symmetry

$$q'^i = q^i(q^i, p_i, \alpha^a), \quad p'_i = p_i(q^i, p_i, \alpha^a), \quad (6)$$

generated by the first class constraints and the finite transformation law for the Lagrange multipliers, the gauge transformed action (1) is computed,

$$S[q'^i, p'_i, \lambda^a] = S[q^i, p_i, \lambda^a] + \Delta S(q^i, p_i, \alpha^a) \big|_{\tau_1}^{\tau_2}. \quad (7)$$

Therefore, the original action (1) can be expressed in terms of the gauge related variables and the gauge parameters

$$S[q^i, p_i, \lambda^a] = S[q'^i, p'_i, \lambda^a] - \Delta S[q^i(q'^i, p'_i, \alpha^a), q_i(q'^i, p'_i, \alpha^a), \alpha^a] \big|_{\tau_1}^{\tau_2}, \quad (8)$$

obtained by plugging into the boundary term ΔS of Eq. (7) the inverse transformation of Eq. (6). At the same time, the original boundary conditions (3) must be rewritten by inserting into the left-hand side (LHS) of Eq. (3) the inverse transformation of Eq. (6),

$$Q^i[q^i(q'^i, p'_i, \alpha^a), p_i(q'^i, p'_i, \alpha^a), \tau] = Q^i_\alpha, \quad \alpha = 1, 2. \quad (9)$$

In this way, the variational principle defined in terms of the gauge related variables, formed with the action (8) and the boundary conditions (9), has a certain freedom encoded in the gauge parameters α^a . The goal is to use this freedom to make compatible both boundary and gauge conditions. In the gauge related variables, the gauge condition is

$$\chi_a(q'^i, p'_i) = 0. \quad (10)$$

Inserting Eq. (6) into Eq. (10) fixes the gauge parameters

$$\alpha^a = \alpha^a(q^i, p_i). \quad (11)$$

The remaining task is to plug Eq. (11) into the right-hand side (RHS) of Eq. (8) and the LHS of Eq. (9) to obtain a well-defined variational principle with both boundary and gauge conditions compatible to each other. Like in the infinitesimal case, $q^i(\tau_\alpha)$ and $p_i(\tau_\alpha)$ play the role of parameters. Of course, the transversality condition has been assumed, which is not necessary if noncanonical gauges are taken into account.

2. Fully gauge-invariant actions. The detailed description of this second type of variational principle, introduced here for the first time, both in its finite and infinitesimal versions follows.

(i) Finite case. Knowing that the action (1) of any generally covariant system transforms in the generic case like Eq. (7), the goal is to build a new action S_{inv} such that it transforms like

$$S_{\text{inv}}[q'^i, p'_i, \lambda^a] \approx S_{\text{inv}}[q^i, p_i, \lambda^a], \quad (12)$$

under the finite gauge transformation. The weak expression (12) means that the gauge invariance of the action is required only until a boundary term proportional to the constraints. It is proposed for S_{inv} the form

$$S_{\text{inv}}[q^i, p_i, \lambda^a] \approx S[q^i, p_i, \lambda^a] - F(q^i, p_i) \Big|_{\tau_1}^{\tau_2} = S_{\text{inv}}[\mathcal{Q}^i, \mathcal{P}_i, \lambda^a], \quad (13)$$

where F is a function of the canonical variables and the last equality is useful to remark that the action (13) is subject to the boundary conditions

$$\mathcal{Q}^i(q^i, p_i, \tau)(\tau_\alpha) = \mathcal{Q}_\alpha^i, \quad (14)$$

determined by Eq. (5) but with B replaced by $B + F$. Of course, it is always possible to add an arbitrary function of Dirac observables O_{phys} , $\{O_{\text{phys}}, G_a\} = O_a{}^b O_{\text{phys}}$ to the time boundary in the RHS of Eq. (13) which would allow the introduction of *ad hoc* boundary conditions.

If the gauge condition is not compatible with Eq. (14), then it can be imposed in the gauge related variables (10), fixing the gauge parameters to Eq. (11). Again, the variational principle is defined in terms of the gauge related variables, its action is simply Eq. (13) but written in terms of the gauge related canonical variables, under the boundary conditions

$$\mathcal{Q}^i(q'^i, p'_i, \tau)(\tau_\alpha) = f^i[\mathcal{Q}_\alpha^i, q^i(\tau_\alpha), p_i(\tau_\alpha), \tau_\alpha], \quad (15)$$

obtained by inserting into the LHS of Eq. (14) both Eq. (11) and the inverse gauge transformation of Eq. (6). Notice that it is not necessary to insert Eq. (11) into Eq. (13) because such action is already gauge-invariant. This is a key difference between the variational principle with non-gauge-invariant actions and the current one. Thus the fixed gauge parameters (11) are inserted just into the boundary conditions. Moreover, Eqs. (14) and (15) have the same functional form in their LHS, the only difference between them is the value they take in their RHS. Again, $q^i(\tau_\alpha)$, and $p_i(\tau_\alpha)$ play the role of parameters. Finally, it is not possible to determine the explicit form for F in the generic case. However, it is possible to write down the differential equations that F must satisfy in the infinitesimal approach.

(ii) Infinitesimal case. Knowing that Eq. (1) transforms infinitesimally like

$$S'[q'^i, p'_i, \lambda'^a] = S[q^i, p_i, \lambda^a] + \left(p_i \frac{\partial G}{\partial p_i} - G - \{B, G\} \right) \Big|_{\tau_1}^{\tau_2}, \quad (16)$$

and F transforms as $F' = F + \{F, G\}$, then F must weakly satisfy the set of differential equations

$$p_i \frac{\partial G}{\partial p_i} \approx \{B + F, G\}, \quad (17)$$

with $G = \epsilon^a G_a$, ϵ^a being the infinitesimal gauge parameters, then in Eq. (17) there are M differential equations for F . Notice the interplay between F and B , in particular $B = 0$ when the configuration variables q^i are fixed at the time boundary (τ_1 and τ_2) in variational principles with non-gauge-invariant actions. In this case, F is completely determined by the explicit form of the first class constraints. If some of them were linear and homogeneous in the momenta then they would not contribute to F , as it happens in Yang-Mills theories. However, if the constraints are quadratic (or higher) in the momenta they will contribute to F . Therefore it is only possible to build fully gauge-invariant actions at least infinitesimally by solving Eq. (17) and is not necessary to add new variables (enlarging the phase space) to build fully gauge-invariant actions, as it is argued in Ref. [6]. The present formalism is completely systematic and could be easily extended to field theory with the obvious modifications and compared with the known results for self-dual gravity [2] and bosonic string theory [7] on this direction. Finally, the issues concerning the transversality condition are those of Ref. [1] for the same reasons that apply in that case.

A. Parametrized relativistic free particle

1. Non-gauge-invariant action

The action for a parametrized relativistic free particle is

$$S[x^\mu, p_\mu, \lambda] = \int_{s_1}^{s_2} ds \left[\frac{dx^\mu}{ds} p_\mu - \lambda (p^\mu p_\mu + m^2 c^2) \right], \quad (18)$$

where the inner product is taken with respect to the Minkowski metric $\eta_{\mu\nu}$ with $\text{diag } \eta_{\mu\nu} = (-1, 1, 1, 1)$, $\mu = 0, 1, 2, 3$, s is an arbitrary parameter (not necessarily the proper time τ) which parametrizes its world line. The standard boundary conditions for Eq. (18) are

$$x^\mu(s_\alpha) = x_\alpha^\mu, \quad \alpha = 1, 2, \quad (19)$$

with x_α^μ prescribed numbers. The constraint

$$G := p^\mu p_\mu + m^2 c^2 = 0 \quad (20)$$

is first class and generates a finite gauge transformation on the phase space variables

$$x'^\mu = x^\mu + \theta(s)p^\mu, \quad p'_\mu = p_\mu, \quad (21)$$

whereas the Lagrange multiplier transforms as

$$\lambda' = \lambda + \frac{1}{2} \frac{d\theta}{ds} \quad (22)$$

with $\theta(s)$ an arbitrary function of s . Now suppose the gauge condition

$$x^0 = 0 \quad (23)$$

were picked. This gauge condition did not match the boundary conditions (19) if $x_\alpha^0 \neq 0$ and then the dynamical problem would be in trouble. This conflict can be solved applying the general scheme already explained. To do this, note that under the gauge symmetry given by Eqs. (21), (22) the action (18) transforms like

$$S[x'^\mu, p'_\mu, \lambda'] = S[x^\mu, p_\mu, \lambda] + \left[(p^\mu p_\mu - m^2 c^2) \frac{\theta}{2} \right]_{s_1}^{s_2}. \quad (24)$$

The boundary term $\Delta S = (p^\mu p_\mu - m^2 c^2) (\theta/2)$ comes from the fact that G is quadratic in the momenta. Notice that ΔS is *not* proportional to G and so S is not gauge-invariant even on the constraint surface.¹ Thus the original action (18) can be expressed in terms of the gauge related variables (x'^μ, p'_μ) , the Lagrange multiplier λ' , and the gauge parameter θ ,

$$S[x^\mu, p_\mu, \lambda] = S[x'^\mu, p'_\mu, \lambda'] - \left[(p'^\mu p'_\mu - m^2 c^2) \frac{\theta}{2} \right]_{s_1}^{s_2}. \quad (25)$$

The RHS of the last expression tells $S[x^\mu, p_\mu, \lambda]$ can be considered a functional of the gauge related variables x'^μ, p'_μ, λ' , and θ . Also, the original boundary conditions (19) must be written in terms of the gauge related variables

$$x'^\mu(s_\alpha) - \theta(s_\alpha)p^\mu(s_\alpha) = x_\alpha^\mu. \quad (26)$$

It is time to set the right dynamical problem with boundary conditions compatible with the gauge condition. In the gauge related variables, the gauge condition is

$$x'^0 = 0. \quad (27)$$

By combining the gauge condition and Eq. (21) the gauge parameter is fixed,

$$\theta = -\frac{x^0}{p^0}. \quad (28)$$

The goal has been reached. The remaining task is to plug into the RHS of Eq. (25) and into the LHS of Eq. (26) last expression for the gauge parameter. By doing this the right action,

$$\begin{aligned} S_{\text{red}}[x'^\mu, p'_\mu, \lambda'](x_\alpha^0, p_\alpha^0) &:= S[x^\mu, p_\mu, \lambda] |_{\theta = -(x^0/p^0)}, \\ &= S[x'^\mu, p'_\mu, \lambda'] + \left[(p'^\mu p'_\mu - m^2 c^2) \frac{1}{2} \frac{x_2^0}{p_2^0} \right] \\ &\quad - \left[(p'^\mu p'_\mu - m^2 c^2) \frac{1}{2} \frac{x_1^0}{p_1^0} \right], \end{aligned} \quad (29)$$

¹ In complex general relativity expressed in terms of Ashtekar variables the boundary term is proportional to the Hamiltonian constraint. See Ref. [2] for the details.

is obtained, under the boundary conditions

$$x'^\mu(s_\alpha) + \frac{x_\alpha^0}{p_\alpha^0} p_\alpha^\mu = x_\alpha^\mu, \quad (30)$$

which are, by construction, compatible with the gauge condition (27). Here $x^\mu(s_\alpha) = x_\alpha^\mu$, and $p^\mu(s_\alpha) = p_\alpha^\mu$ which play the role of “parameters” in the final action (29).² In summary, the analysis began with a wrong variational principle where the boundary conditions were not compatible with the gauge condition, and a right variational principle with boundary conditions compatible with the gauge condition was built. In the new variational principle: (i) a new action including a boundary term was constructed [see Eq. (29)]; (ii) the boundary conditions were also modified [see Eq. (30)].

2. Fully gauge-invariant action

Now, the original action will be fully gauge-invariant under the gauge symmetry generated by the constraint G *in spite of the fact* the constraint G is quadratic in the momenta. The simplest boundary term needed to build S_{inv} is $F = x^\mu p_\mu$,

$$S_{\text{inv}}[x^\mu, p_\mu, \lambda] = \int_{s_1}^{s_2} ds \left[\frac{dx^\mu}{ds} p_\mu - \lambda (p^\mu p_\mu + m^2 c^2) \right] - x^\mu p_\mu \Big|_{s_1}^{s_2}. \quad (31)$$

In fact, by using the finite gauge transformation (21) and (22),

$$S_{\text{inv}}[x'^\mu, p'_\mu, \lambda'] = S_{\text{inv}}[x^\mu, p_\mu, \lambda] - \frac{\theta}{2} (p^\mu p_\mu + m^2 c^2) \Big|_{s_1}^{s_2}, \quad (32)$$

and the difference between $S_{\text{inv}}[x'^\mu, p'_\mu, \lambda']$ and $S_{\text{inv}}[x^\mu, p_\mu, \lambda]$ is a boundary term which is *proportional* to the first class constraint G . Therefore S_{inv} is gauge-invariant on the constraint surface $G = 0$ only. Of course an arbitrary function of the Dirac observables for the system $F_1(p_\mu, x^\mu p^\mu - x^\nu p^\nu)$ might have been (and can be) added to the time boundary of Eq. (31) too without destroying the gauge invariance of S_{inv} , just modifying the boundary conditions.

To find the new boundary conditions associated with the action (31) the canonical transformation induced by its boundary term will be used. By using Eq. (5) the new phase space variables,

$$\mathcal{Q}^\mu = -\frac{1}{\beta} p^\mu, \quad \mathcal{P}_\mu = \beta x_\mu, \quad (33)$$

are obtained, with β a nonvanishing constant. Thus the boundary conditions associated with Eq. (31) are

$$\mathcal{Q}^\mu(s_\alpha) = -\frac{1}{\beta} p^\mu(s_\alpha) = \mathcal{Q}_\alpha^\mu. \quad (34)$$

It is immediately seen that the wanted gauge condition $x^0 = 0$ does not conflict with these boundary conditions.

B. Parametrized harmonic oscillator

1. Non-gauge-invariant action

In this case, the original variational principle is defined by the action

$$S[x, t, p, p_t, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dx}{d\tau} p + \frac{dt}{d\tau} p_t - \lambda \left(p_t + \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \right) \right], \quad (35)$$

² From now on Kuchar’s notation is used, namely, $S\cdots$ is a functional of the variables inside the square brackets and a function of the variables inside the parentheses.

under the standard boundary conditions

$$x(\tau_\alpha) = x_\alpha, \quad t(\tau_\alpha) = t_\alpha, \quad \alpha = 1, 2, \quad (36)$$

with x_α and t_α prescribed numbers. The constraint

$$G := p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = 0, \quad (37)$$

is first class and generates a *finite* gauge transformation on the phase space variables [10],

$$\begin{aligned} x' &= x \cos \theta(\tau) + \frac{p}{m\omega} \sin \theta(\tau), \\ p' &= -m\omega x \sin \theta(\tau) + p \cos \theta(\tau), \\ t' &= \frac{\theta(\tau)}{\omega} + t, \\ p'_t &= p_t, \end{aligned} \quad (38)$$

whereas the Lagrange multiplier transforms as

$$\lambda' = \lambda + \frac{\dot{\theta}(\tau)}{\omega}, \quad (39)$$

with $\dot{\theta}(\tau) = d\theta(\tau)/d\tau$.

Again, suppose the gauge condition

$$t = 0 \quad (40)$$

were picked. Obviously, it did not match the boundary conditions (36) if $t_\alpha \neq 0$. It is time to apply the method. By using the *finite* gauge transformation (38), (39) the action (35) transforms like

$$S[x', t', p', p'_t, \lambda'] = S[x, t, p, p_t, \lambda] + \left[-\sin^2 \theta x p + \frac{1}{\omega} \left(\frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 \right) \frac{\sin 2\theta}{2} \right]_{\tau_1}^{\tau_2}. \quad (41)$$

Again, the boundary term $\Delta S = -\sin^2 \theta x p + (1/\omega) \left(\frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 \right) (\sin 2\theta)/2$ comes from the quadratic in the momenta term of G . Note that ΔS is *not* proportional to G . Therefore

$$S[x, t, p, p_t, \lambda] = S[x', t', p', p'_t, \lambda'] - \left[\sin^2 \theta x' p' + \frac{1}{\omega} \left(\frac{p'^2}{2m} - \frac{1}{2}m\omega^2 x'^2 \right) \frac{\sin 2\theta}{2} \right]_{\tau_1}^{\tau_2}. \quad (42)$$

From the RHS of the last equation it is clear that $S[x, t, p, p_t, \lambda]$ can be considered a functional of $x', t', p', p'_t, \lambda'$, and θ . At the same time, the *original* boundary conditions (36) must be written in terms of the gauge related variables and the gauge parameter θ ,

$$\begin{aligned} x'(\tau_\alpha) \cos \theta(\tau_\alpha) - \frac{p'(\tau_\alpha)}{m\omega} \sin \theta(\tau_\alpha) &= x_\alpha, \\ t'(\tau_\alpha) - \frac{\theta(\tau_\alpha)}{\omega} &= t_\alpha, \quad \alpha = 1, 2. \end{aligned} \quad (43)$$

It is time to define the new variational principle whose boundary conditions will be compatible with the required gauge condition. In the gauge related variables, the gauge condition is

$$t' = 0. \quad (44)$$

By using the transformation law for the t variable (38) together with the required gauge condition (44) the explicit expression for the gauge parameter is obtained, $\theta = -\omega t$. The goal has been reached. A right variational principle with canonical pairs (x', p') and (t', p'_t) can be built, its action is

$$\begin{aligned}
S_{\text{red}}[x', t', p', p'_t, \lambda'](t_1, t_2) &:= S[x, t, p, p_t, \lambda] |_{\theta=-\omega t}, \\
&= S[x', t', p', p'_t, \lambda'] \\
&\quad - \left[\sin^2 \omega t_2 x'(\tau_2) p'(\tau_2) - \frac{1}{\omega} \left(\frac{p'^2(\tau_2)}{2m} - \frac{1}{2} m \omega^2 x'^2(\tau_2) \right) \frac{\sin 2\omega t_2}{2} \right] \\
&\quad + \left[\sin^2 \omega t_1 x'(\tau_1) p'(\tau_1) - \frac{1}{\omega} \left(\frac{p'^2(\tau_1)}{2m} - \frac{1}{2} m \omega^2 x'^2(\tau_1) \right) \frac{\sin 2\omega t_1}{2} \right],
\end{aligned} \tag{45}$$

under the boundary conditions

$$\begin{aligned}
x'(\tau_\alpha) \cos \omega t_\alpha + \frac{p'(\tau_\alpha)}{m\omega} \sin \omega t_\alpha &= x_\alpha, \\
t'(\tau_\alpha) &= 0, \quad \alpha = 1, 2,
\end{aligned} \tag{46}$$

which are, by construction, compatible with the gauge condition

$$t' = 0. \tag{47}$$

Once the dynamical problem has been well defined, there are still two remaining things to do. The first one is to compute the gauge-fixed variational principle by plugging into the action $S_{\text{red}}[x', t', p', p'_t, \lambda'](t_1, t_2)$ and into the boundary conditions (46) the gauge condition $t' = 0$ as well as the constraint $G' = 0$. The second one is to solve the dynamics by using the equations of motion with the gauge condition $t' = 0$. Let us focus in the first option. By plugging the gauge condition $t' = 0$ and the constraint $G' = 0$ into the action $S_{\text{red}}[x', t', p', p'_t, \lambda'](t_1, t_2)$, the gauge-fixed action

$$\begin{aligned}
S_{\text{fixed}}[x', p'](t_1, t_2) &= S_{\text{red}}[x, t, p', p'_t, \lambda'](t_1, t_2) |_{G'=0, t'=0}, \\
&= S[x, t, p, p_t, \lambda]_{\theta=-\omega t, G'=0, t'=0}, \\
&= \int_{\tau_1}^{\tau_2} d\tau \frac{dx'}{d\tau} p' \\
&\quad - \left[\sin^2 \omega t_2 x'(\tau_2) p'(\tau_2) - \frac{1}{\omega} \left(\frac{p'^2(\tau_2)}{2m} - \frac{1}{2} m \omega^2 x'^2(\tau_2) \right) \frac{\sin 2\omega t_2}{2} \right] \\
&\quad + \left[\sin^2 \omega t_1 x'(\tau_1) p'(\tau_1) - \frac{1}{\omega} \left(\frac{p'^2(\tau_1)}{2m} - \frac{1}{2} m \omega^2 x'^2(\tau_1) \right) \frac{\sin 2\omega t_1}{2} \right].
\end{aligned} \tag{48}$$

is obtained. This form of the action is very interesting. It contains a kinematical term like any action, but it has not a Hamiltonian, rather, all its dynamical information has been mapped to its time boundary. Therefore it is natural to interpret this result as the “canonical version” of the holographic hypothesis in the sense that with this particular choice of the gauge condition, its dynamics is now at the time boundary [8]. In fact, there was a “reduction” of degrees of freedom, before fixing the gauge the initial dynamical problem was defined on the closed set $[\tau_1, \tau_2]$ whereas the final dynamical problem is now defined on just *two* points, τ_2 and τ_1 . Of course, the gauge-fixed variational principle has associated the remaining boundary conditions,

$$x'(\tau_\alpha) \cos \omega t_\alpha + \frac{p'(\tau_\alpha)}{m\omega} \sin \omega t_\alpha = x_\alpha, \quad \alpha = 1, 2. \tag{49}$$

Note that the number of boundary conditions has decreased.

Alternatively, the second option is to solve the equations of motion which are the original ones but with x, t, p, p_t , and λ replaced by x', t', p', p'_t , and λ' . Their solution, using the gauge $t' = 0$, is

$$\begin{aligned}
x' &= x_0, \\
p' &= p_0, \\
t' &= 0, \\
p'_t &= -\frac{p_0^2}{2m} - \frac{1}{2} m \omega^2 x_0^2,
\end{aligned} \tag{50}$$

with x_0, p_0 constants (τ independent), and therefore they are Dirac observables. Inserting this solution into the action (45), the term $S[x', t', p', p'_t, \lambda']$ vanishes, and the only contribution is given by the boundary term

$$S(x_0, p_0; t_1, t_2) = - \left[\sin^2 \omega t_2 x_0 p_0 - \frac{1}{\omega} \left(\frac{p_0^2}{2m} - \frac{1}{2} m \omega^2 x_0^2 \right) \frac{\sin 2\omega t_2}{2} \right] + \left[\sin^2 \omega t_1 x_0 p_0 - \frac{1}{\omega} \left(\frac{p_0^2}{2m} - \frac{1}{2} m \omega^2 x_0^2 \right) \frac{\sin 2\omega t_1}{2} \right], \quad (51)$$

where $x'(\tau_\alpha) = x_0$ and $p'(\tau_\alpha) = p_0$ were used. It is clear that $S(x_0, p_0; t_1, t_2)$ represents a two parameter family of physical (Dirac) observables on the reduced phase space labeled by x_0 and p_0 ; t_1 and t_2 being the parameters. Of course, Eqs. (48) and (51) are the same thing but they look different because it has not been inserted in Eq. (48) the fact-coming from the equations of motion-that $x' = x_0$ and $p' = p_0$. Using this information, Eq. (48) acquires the form (51). In addition, the boundary conditions, of course, reduce to

$$x_0 \cos \omega t_\alpha + \frac{p_0}{m\omega} \sin \omega t_\alpha = x_\alpha, \quad \alpha = 1, 2, \quad (52)$$

establishing a relationship between the initial and final data $[x_1, x_2, t_1, \text{ and } t_2]$ and the physical states (x_0 , and p_0) in the reduced phase space, displaying the fact that dynamics of the parametrized harmonic oscillator between τ_1 and τ_2 is pure gauge, namely, it corresponds to a point (x_0, p_0) in the reduced phase space for each set of initial and final data (x_1, x_2, t_1, t_2) .

To find the explicit expressions for these observables in terms of the original phase space variables (x, p) , and (t, p_t) , it is necessary to use the relationship between the original phase space variables (x, t, p, p_t) and the gauge related ones (x', t', p', p'_t) together with the expression for the gauge parameter $\theta = -\omega t$. From them,

$$\begin{aligned} x' &= x \cos \omega t - \frac{p}{m\omega} \sin \omega t, \\ p' &= m\omega x \sin \omega t + p \cos \omega t. \end{aligned} \quad (53)$$

But, because of Eq. (50), these two expressions for x' and p' must be the same thing. Therefore x_0 and p_0 are given by [10]

$$\begin{aligned} x_0 &= x \cos \omega t - \frac{p}{m\omega} \sin \omega t, \\ p_0 &= m\omega x \sin \omega t + p \cos \omega t, \end{aligned} \quad (54)$$

satisfying $\{x_0, p_0\} = 1$. Computing Eq. (54) at $t = 0$, it can be shown that these variables correspond to the initial conditions. Furthermore, considering the inverse transformation of Eq. (54) it follows that dynamics of the system is expressed in terms of the initial conditions.

2. Fully gauge-invariant action

Now, the original action will be fully gauge-invariant under the gauge symmetry generated by the first class constraint G in spite of the fact that the constraint G is quadratic in the momentum p , $p^2/2m$. The simplest boundary term needed to build S_{inv} is $F = -\frac{1}{2}xp$,

$$S_{\text{inv}}[x, t, p, p_t, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dx}{d\tau} p + \frac{dt}{d\tau} p_t - \lambda \left(p_t + \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \right) \right] - \frac{1}{2} xp \Big|_{\tau_1}^{\tau_2}. \quad (55)$$

Using Eqs. (38) and (39), it is clear that

$$S_{\text{inv}}[x', t', p', p'_t, \lambda'] = S_{\text{inv}}[x, t, p, p_t, \lambda]. \quad (56)$$

Thus the action S_{inv} is indeed fully gauge-invariant under the finite gauge symmetry involved. Again, an arbitrary function $F_1(p^2/2m + \frac{1}{2}m\omega^2 x^2)$ might have been (and can be) added to the time boundary of Eq. (55). However, Eq. (55) is the simplest form. The boundary term in Eq. (55) induces the canonical transformation (5) from the original set of variables (x, t, p, p_t) to the new set $(\mathcal{X}, \mathcal{T}, \mathcal{P}, \mathcal{P}_\mathcal{T})$,

$$\mathcal{X} = \frac{1}{2} \ln \left(\frac{x}{p} \right), \quad \mathcal{P} = xp, \quad \mathcal{T} = t, \quad \mathcal{P}_\mathcal{T} = p_t. \quad (57)$$

Therefore the boundary conditions associated with the action S_{inv} are

$$\frac{1}{2} \ln \left(\frac{x}{p} \right) (\tau_\alpha) = \mathcal{X}_\alpha, \quad t(\tau_\alpha) = \mathcal{T}_\alpha, \quad \alpha = 1, 2, \quad (58)$$

where a dimensional constant equals to 1 is understood. Again, suppose the gauge condition

$$t = 0 \quad (59)$$

were picked, then it is pretty obvious that it would conflict the boundary conditions (58) provided $\mathcal{T}_\alpha \neq 0$. Nevertheless, the gauge condition can be reached in the gauge-related variables, namely,

$$t' = 0. \quad (60)$$

By applying the method, the right action is

$$S_{\text{inv}}[x', t', p', p'_t, \lambda'] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dx'}{d\tau} p' + \frac{dt'}{d\tau} p'_t - \lambda' \left(p'_t + \frac{p'^2}{2m} + \frac{1}{2} m \omega^2 x'^2 \right) \right] - \frac{1}{2} x' p' \Big|_{\tau_1}^{\tau_2}, \quad (61)$$

under the boundary conditions

$$\begin{aligned} \frac{1}{2} \ln \left(\frac{x'}{p'} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{e^{2\mathcal{X}_\alpha} \cos \omega \mathcal{T}_\alpha - \frac{1}{m\omega} \sin \omega \mathcal{T}_\alpha}{e^{2\mathcal{X}_\alpha} \sin \omega \mathcal{T}_\alpha + \cos \omega \mathcal{T}_\alpha} \right), \\ t'(\tau_\alpha) &= 0, \quad \alpha = 1, 2, \end{aligned} \quad (62)$$

which are, by construction, compatible with the gauge condition (60). Therefore this variational principle based on a fully gauge-invariant action is more “economic” than the one based on a non-gauge-invariant action because in the former it is *not* necessary to handle additional boundary terms for the original action is already fully gauge-invariant. Also, the boundary conditions in terms of the original variables and in terms of the gauge related ones look more “symmetric” [see Eqs. (58) and (62)], the difference between them being the value they take in their RHS.

Again, once the gauge conditions have been made compatible with the boundary conditions, there are still two remaining things to do. The first one is to compute the gauge-fixed variational principle by plugging into the action and into the boundary conditions the gauge condition $t' = 0$ together with the constraint $G' = 0$. By doing this

$$\begin{aligned} S_{\text{fixed}}[x', p'] &= S_{\text{inv}}[x', t', p', p'_t, \lambda'] \Big|_{G'=0, t'=0}, \\ &= \int_{\tau_1}^{\tau_2} d\tau \frac{dx'}{d\tau} p' - \frac{1}{2} x' p' \Big|_{\tau_1}^{\tau_2}, \end{aligned} \quad (63)$$

is the gauge-fixed action and its boundary conditions are

$$\frac{1}{2} \ln \left(\frac{x'}{p'} \right) (\tau_\alpha) = \frac{1}{2} \ln \left(\frac{e^{2\mathcal{X}_\alpha} \cos \omega \mathcal{T}_\alpha - \frac{1}{m\omega} \sin \omega \mathcal{T}_\alpha}{e^{2\mathcal{X}_\alpha} \sin \omega \mathcal{T}_\alpha + \cos \omega \mathcal{T}_\alpha} \right), \quad \alpha = 1, 2. \quad (64)$$

This form of the action contains a kinetic term, as any action, and it has not a Hamiltonian. Where is the dynamics contained? Obviously it is fully contained in the boundary terms, in a similar way to Eq. (54). Again, this result might be interpreted as an implementation of the holographic hypothesis in the sense dynamics has been mapped to its time boundary [8].

Finally, it can be easily checked that the dynamics coming from the last variational principle is the same as the one coming from the equations of motion when the gauge condition $t' = 0$ is imposed, as was done in the noninvariant case discussed in the previous subsection. It makes no sense to repeat this computation.

C. $SL(2, R)$ model

1. Non-gauge-invariant $SL(2, R)$ model

Up to here, generally covariant systems with a single Hamiltonian constraint have been studied. Next, a model with two noncommuting Hamiltonian constraints and one constraint linear and homogeneous in the momenta will

be analyzed. The nontrivial interplay among linear and quadratic constraints will produce a contribution to the boundary term of the linear ones in direct opposition to what the infinitesimal approach says.

This model has a $SL(2, R)$ gauge symmetry, one continuous physical degree of freedom, and mimics the constraint structure

$$\{H, H\} \sim D, \quad \{H, D\} \sim H, \quad \{D, D\} \sim D, \quad (65)$$

of general relativity. It can be considered as a (two points) discrete version of the Arnowitt-Deser-Misner (ADM) formulation of gravity. Readers are urged to read Ref. [3] for the details, in particular for a clear description of the relational evolution of the degrees of freedom of the system (on this see also Refs. [9, 10, 11]). Its Hamiltonian action is

$$S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{\vec{u}} \cdot \vec{p} + \dot{\vec{v}} \cdot \vec{\pi} - (NH_1 + MH_2 + \lambda D) \right], \quad (66)$$

under the boundary conditions

$$\vec{u}(\tau_\alpha) = \vec{U}_\alpha, \quad \vec{v}(\tau_\alpha) = \vec{V}_\alpha, \quad \alpha = 1, 2. \quad (67)$$

The canonical pairs are (\vec{u}, \vec{p}) , and $(\vec{v}, \vec{\pi})$; each vector being a two-dimensional real one, the scalar product is taken in E^2 . N , M , and λ are Lagrange multipliers. The constraints H_1 , H_2 , and D ,

$$\begin{aligned} H_1 &:= \frac{1}{2}(\vec{p}^2 - \vec{v}^2) = 0, \\ H_2 &:= \frac{1}{2}(\vec{\pi}^2 - \vec{u}^2) = 0, \\ D &:= \vec{u} \cdot \vec{p} - \vec{v} \cdot \vec{\pi} = 0, \end{aligned} \quad (68)$$

are first class, with $\vec{p}^2 = \vec{p} \cdot \vec{p} = (p_1)^2 + (p_2)^2$, and so on. The constraint algebra is isomorphic to the $sl(2, R)$ Lie algebra and the *finite* gauge transformation the constraints generate is [3]

$$\begin{aligned} \vec{u}' &= \alpha(\tau)\vec{u} + \beta(\tau)\vec{p}, & \vec{\pi}' &= \alpha(\tau)\vec{\pi} + \beta(\tau)\vec{v}, \\ \vec{p}' &= \gamma(\tau)\vec{u} + \delta(\tau)\vec{p}, & \vec{v}' &= \gamma(\tau)\vec{\pi} + \delta(\tau)\vec{v}, \end{aligned} \quad (69)$$

where the matrix

$$G(\tau) = \begin{pmatrix} \alpha(\tau) & \beta(\tau) \\ \gamma(\tau) & \delta(\tau) \end{pmatrix} \quad (70)$$

belongs to the $SL(2, R)$ group, i.e., it satisfies $\alpha(\tau)\delta(\tau) - \beta(\tau)\gamma(\tau) = 1$. So, the system is invariant under a $SL(2, R)$ gauge symmetry local in τ . The finite transformation law for the Lagrange multipliers is

$$\begin{aligned} \begin{pmatrix} \lambda' & N' \\ M' & -\lambda' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda & N \\ M & -\lambda \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \\ &\quad - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \dot{\delta} & -\dot{\beta} \\ -\dot{\gamma} & \dot{\alpha} \end{pmatrix}, \end{aligned} \quad (71)$$

so, the matrix (71) transforms as a Yang-Mills connection valued in the Lie algebra of $SL(2, R)$ [3].

Now, suppose the gauge condition

$$u^1 = A, \quad u^2 = B, \quad p_1 = C \quad (72)$$

were picked. Obviously it did not match with the boundary conditions (67) in general. Under the *finite* gauge transformation (69) and (71) the change of the action (66) is

$$\begin{aligned} S[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] &= S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] \\ &\quad + \left[(\beta\gamma)(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi}) + \frac{1}{2}(\alpha\gamma)(\vec{u}^2 + \vec{\pi}^2) + \frac{1}{2}(\beta\delta)(\vec{v}^2 + \vec{p}^2) \right]_{\tau_1}^{\tau_2}. \end{aligned} \quad (73)$$

The boundary term $\Delta S = (\beta\gamma)(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi}) + \frac{1}{2}(\alpha\gamma)(\vec{u}^2 + \vec{\pi}^2) + \frac{1}{2}(\beta\delta)(\vec{v}^2 + \vec{p}^2)$ comes from the two noncommuting Hamiltonian constraints. Also, ΔS is *not* a linear combination of the first class constraints. More important, the

term $(\beta\gamma)(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi})$ is *not* present at the infinitesimal level. This term is associated with the “diffeomorphism” constraint D , which is linear and homogeneous in the momenta and so it does not contribute at the *infinitesimal* level. Nevertheless, when *finite* gauge transformations are taken into account, the contribution associated with this constraint appears again. Notice that if the two Hamiltonian constraints were turned off, namely, $\beta = 0$ and $\gamma = 0$, then the action would be gauge-invariant as expected because the only remaining constraint would be D , which is linear and homogeneous in the momenta. Here, it will be taken into account the full $SL(2, R)$ gauge symmetry and not just only a subgroup of it. By using again the finite gauge transformation (69) and (71) the original action can be written as

$$S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] = S[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] - \left[-(\beta\gamma)(\vec{u}' \cdot \vec{p}' + \vec{v}' \cdot \vec{\pi}') + \frac{1}{2}(\gamma\delta)(\vec{u}'^2 + \vec{\pi}'^2) + \frac{1}{2}(\alpha\beta)(\vec{v}'^2 + \vec{p}'^2) \right]_{\tau_1}^{\tau_2}. \quad (74)$$

From the RHS of last expression it is clear that $S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda]$ can be interpreted as a functional of $\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda', \alpha, \beta, \gamma$, and δ . In the same way, it is possible to rewrite the *original* boundary conditions (67) in terms of the gauge related variables as well as of the gauge parameters α, β, γ , and δ ,

$$\begin{aligned} \delta(\tau_\alpha)u'^i(\tau_\alpha) - \beta(\tau_\alpha)p'_i(\tau_\alpha) &= U_\alpha^i, \\ \alpha(\tau_\alpha)v'^i(\tau_\alpha) - \gamma(\tau_\alpha)\pi'_i(\tau_\alpha) &= V_\alpha^i, \quad \alpha = 1, 2; \quad i = 1, 2. \end{aligned} \quad (75)$$

It is time to define the new variational principle whose boundary conditions will be compatible with the required gauge condition. This new variational principle is defined in terms of the gauge related variables $(\vec{u}', \vec{v}', \vec{p}', \vec{\pi}')$, given by the RHS side of Eq. (74), and its boundary conditions will be those given in Eq. (75). In the gauge related variables, the gauge condition is

$$u'^1 = A, \quad u'^2 = B, \quad p'_1 = C. \quad (76)$$

The explicit expressions for the gauge parameters are computed using Eqs. (69), (76) together with $\alpha\delta - \beta\gamma = 1$,

$$\begin{aligned} \alpha &= \frac{Ap_2 - Bp_1}{O_{12}}, \\ \beta &= \frac{Bu^1 - Au^2}{O_{12}}, \\ \gamma &= \frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}}, \\ \delta &= \frac{-ACu^2 + (BC + O_{12})u^1}{AO_{12}}, \end{aligned} \quad (77)$$

where $O_{12} = u^1p_2 - u^2p_1$ is a physical observable [3]. The goal has been reached, i.e., it has been possible to build a variational principle where the canonical pairs are (\vec{u}', \vec{p}') and $(\vec{v}', \vec{\pi}')$, its action is given by

$$\begin{aligned} S_{\text{red}}[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] &:= S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] |_{\alpha, \beta, \gamma, \delta}, \\ &= S[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] \\ &\quad + \left[\left(\frac{Bu^1 - Au^2}{O_{12}} \right) \left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) (\vec{u}' \cdot \vec{p}' + \vec{v}' \cdot \vec{\pi}') \right]_{\tau_1}^{\tau_2} \\ &\quad - \left[\frac{1}{2} \left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) \left(\frac{-ACu^2 + (BC + O_{12})u^1}{AO_{12}} \right) (\vec{u}'^2 + \vec{\pi}'^2) \right]_{\tau_1}^{\tau_2} \\ &\quad - \left[\frac{1}{2} \left(\frac{Ap_2 - Bp_1}{O_{12}} \right) \left(\frac{Bu^1 - Au^2}{O_{12}} \right) (\vec{v}'^2 + \vec{p}'^2) \right]_{\tau_1}^{\tau_2} \end{aligned} \quad (78)$$

and the boundary conditions are

$$\begin{aligned} \left(\frac{-ACu^2 + (BC + O_{12})u^1}{AO_{12}} \right) (\tau_\alpha) u'^i(\tau_\alpha) - \left(\frac{Bu^1 - Au^2}{O_{12}} \right) (\tau_\alpha) p'_i(\tau_\alpha) &= U_\alpha^i, \\ - \left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) (\tau_\alpha) \pi'_i(\tau_\alpha) + \left(\frac{Ap_2 - Bp_1}{O_{12}} \right) (\tau_\alpha) v'^i(\tau_\alpha) &= V_\alpha^i, \quad \alpha = 1, 2; \quad i = 1, 2, \end{aligned} \quad (79)$$

which are, by construction, compatible with the gauge condition (76).

Once the dynamical problem has been well defined, there are still two remaining things to do. The first one is to compute the gauge-fixed variational principle by plugging into both the action $S_{\text{red}}[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] (u^i(\tau_\alpha), p_i(\tau_\alpha))$ and into the boundary conditions (79) the gauge condition (76) together with the first class constraints equal to zero. The second one is to solve the dynamics by using the equations of motion and the gauge condition (76). Let us focus on the first option. By plugging the gauge condition (76) and the first class constraints equal to zero into the action $S_{\text{red}}[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] (u^i(\tau_\alpha), p_i(\tau_\alpha))$, the gauge-fixed action

$$\begin{aligned}
& S_{\text{fixed}}[\vec{v}', \vec{\pi}'] (u^i(\tau_\alpha)) = \\
& = S_{\text{red}}[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] (u^i(\tau_\alpha), p_i(\tau_\alpha)) \big|_{\alpha, \beta, \gamma, \delta; H'_1=0, H'_2=0, D'=0, u'^1=A, u'^2=B, p'_1=C}, \\
& = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dv'^1}{d\tau} \pi'_1 + \frac{dv'^2}{d\tau} \pi'_2 \right] \\
& + \left[\left(\frac{Bu^1 - Au^2}{O_{12}} \right) \left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) (2\vec{v}' \cdot \vec{\pi}') \right]_{\tau_1}^{\tau_2} \\
& - \left[\left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) \left(\frac{-ACu^2 + (BC + O_{12})u^1}{AO_{12}} \right) \vec{\pi}'^2 \right]_{\tau_1}^{\tau_2} \\
& - \left[\left(\frac{Ap_2 - Bp_1}{O_{12}} \right) \left(\frac{Bu^1 - Au^2}{O_{12}} \right) \vec{v}'^2 \right]_{\tau_1}^{\tau_2}, \tag{80}
\end{aligned}$$

is obtained, under the boundary conditions

$$- \left(\frac{ACp_2 - (BC + O_{12})p_1}{AO_{12}} \right) (\tau_\alpha) \pi'_i(\tau_\alpha) + \left(\frac{Ap_2 - Bp_1}{O_{12}} \right) (\tau_\alpha) v'^i(\tau_\alpha) = V_\alpha^i, \quad \alpha = 1, 2; \quad i = 1, 2. \tag{81}$$

In this variational principle the phase space variables are $(\vec{v}', \vec{\pi}')$ and $u^i(\tau_\alpha) = U_\alpha^i$ are parameters [cf Eq. (48) in the case of the harmonic oscillator]. Note that in the boundary term of last action as well as in last boundary conditions $p_i(\tau_\alpha)$ is a function of $A, B, C, u^i(\tau_\alpha), v'^i(\tau_\alpha)$, and $p'_2(\tau_\alpha)$. Therefore the boundary conditions and the action are well defined. To arrive at this result the first set of equations in Eq. (79) was used, which gives $p_i(\tau_\alpha)$ as a function of A, B, C , and $p'_2(\tau_\alpha)$. Nevertheless, using the constraint $D' = 0$, $p'_2(\tau_\alpha)$ can be put as a function of $A, B, C, v'^i(\tau_\alpha)$, and $\pi'_i(\tau_\alpha)$. The final result comes from the combination of these two partial results. Notice also that the number of boundary conditions has decreased. This form of the variational principle is very interesting. It contains a kinematical term like any action, but it has not a Hamiltonian, that is to say, all its dynamical information has been mapped to its time boundary. Therefore it is natural to interpret this result as the “canonical version” of the holographic hypothesis in the sense that with this particular choice of the gauge condition, the dynamics of the system is now at the time boundary [8].

Alternatively, the second option is to solve the equations of motion which are the original ones but with $\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M$, and λ replaced by $\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M'$, and λ' . Their solution, using the gauge $u'^1 = A, u'^2 = B, p'_1 = C$ is

$$\begin{aligned}
u'^1 &= A, & u'^2 &= B, & p'_1 &= C, & p'^2 &= D, \\
v'^1 &= E, & v'^2 &= F, & \pi'_1 &= G, & \pi'_2 &= H.
\end{aligned} \tag{82}$$

with $A \cdots H$ constants (τ independent), and therefore they are Dirac observables (of course they are not independent, rather, they are related by means of the constraint equations). Inserting this solution in the action (78), the term $S[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda']$ vanishes, and the only contribution is given by the boundary term there.

It is worth emphasizing that $D \cdots H$ are indeed Dirac observables. To obtain the explicit expressions of these observables in terms of the original phase space variables $\vec{u}, \vec{v}, \vec{p}$, and $\vec{\pi}$, it is necessary to use Eqs. (69) and (77). From them,

$$\begin{aligned}
p'_2 &= \frac{O_{12} + AC}{A}, \\
\pi'_1 &= \frac{-AO_{23} + BO_{13}}{O_{12}}, \\
\pi'_2 &= \frac{-AO_{24} + BO_{14}}{O_{12}},
\end{aligned}$$

$$\begin{aligned}
v'^1 &= \frac{-ACO_{23} + O_{13}(O_{12} + BC)}{AO_{12}}, \\
v'^2 &= \frac{-ACO_{24} + O_{14}(O_{12} + BC)}{AO_{12}}
\end{aligned} \tag{83}$$

(see Ref. [3] for the definition of the O_{ij} observables). But, because of Eq. (82), these two expressions for \vec{u}' , \vec{v}' , \vec{p}' , and $\vec{\pi}'$ must be the same thing. Therefore D , E , F , G , and H are given by

$$\begin{aligned}
D &= \frac{O_{12} + AC}{A}, \\
E &= \frac{-AO_{23} + BO_{13}}{O_{12}}, \\
F &= \frac{-AO_{24} + BO_{14}}{O_{12}}, \\
G &= \frac{-ACO_{23} + O_{13}(O_{12} + BC)}{AO_{12}}, \\
H &= \frac{-ACO_{24} + O_{14}(O_{12} + BC)}{AO_{12}}.
\end{aligned} \tag{84}$$

Of course, these five observables are non independent, there are restrictions among them [3]. The important point is that this shows that the dynamics is frozen in this particular gauge.

2. Fully gauge-invariant $SL(2, R)$ model

The simplest boundary term needed to build S_{inv} is $F = \frac{1}{2}(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi})$. Of course an arbitrary function of the Dirac observables $F_1(\phi, J, \epsilon, \epsilon')$ can be added too. However, particular choices for F_1 just modify the boundary conditions. Therefore the simplest variational principle has the gauge-invariant action

$$S_{\text{inv}}[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] = S - \frac{1}{2}(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi})|_{\tau_1}^{\tau_2}. \tag{85}$$

Due to the fact that the action S has been replaced by S_{inv} , the boundary conditions must be modified too. Using Eq. (5), the canonical transformation induced by the boundary term B_2 is

$$\begin{aligned}
Q^1 &= \frac{1}{2} \ln \left(\frac{u^1}{p_1} \right), & P_1 &= u^1 p_1, \\
Q^2 &= \frac{1}{2} \ln \left(\frac{u^2}{p_2} \right), & P_2 &= u^2 p_2, \\
Q^3 &= \frac{1}{2} \ln \left(\frac{v^1}{\pi_1} \right), & P_3 &= v^1 \pi_1, \\
Q^4 &= \frac{1}{2} \ln \left(\frac{v^2}{\pi_2} \right), & P_4 &= v^2 \pi_2.
\end{aligned} \tag{86}$$

Thus the new boundary conditions associated with S_{inv} are

$$\begin{aligned}
\frac{1}{2} \ln \left(\frac{u^1}{p_1} \right) (\tau_\alpha) &= Q_\alpha^1, \\
\frac{1}{2} \ln \left(\frac{u^2}{p_2} \right) (\tau_\alpha) &= Q_\alpha^2, \\
\frac{1}{2} \ln \left(\frac{v^1}{\pi_1} \right) (\tau_\alpha) &= Q_\alpha^3, \\
\frac{1}{2} \ln \left(\frac{v^2}{\pi_2} \right) (\tau_\alpha) &= Q_\alpha^4, \quad \alpha = 1, 2.
\end{aligned} \tag{87}$$

In summary, the original variational principle is defined by the action (85) and by the boundary conditions (87) if (\vec{u}, \vec{p}) , and $(\vec{v}, \vec{\pi})$ are used as canonical variables. Suppose the boundary conditions

$$u^1 = A, \quad u^2 = B, \quad p_1 = C \tag{88}$$

were imposed. It is clear that they would conflict with the boundary conditions. Applying the method, the action in the new variational principle is simply Eq. (85) but rewritten in terms of (\vec{u}', \vec{p}') , and $(\vec{v}', \vec{\pi}')$

$$\begin{aligned} S_{\text{inv}}[\vec{u}', \vec{v}', \vec{p}', \vec{\pi}', N', M', \lambda'] &= S' - B_2' \big|_{\tau_1}^{\tau_2}, \\ &= S' - \frac{1}{2} (\vec{u}' \cdot \vec{p}' + \vec{v}' \cdot \vec{\pi}') \big|_{\tau_1}^{\tau_2}, \end{aligned} \quad (89)$$

under the boundary conditions

$$\begin{aligned} \frac{1}{2} \ln \left(\frac{u'^1}{p'_1} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\beta(\tau_\alpha) + \alpha(\tau_\alpha) e^{2\mathcal{Q}_\alpha^1}}{\delta(\tau_\alpha) + \gamma(\tau_\alpha) e^{2\mathcal{Q}_\alpha^1}} \right), \\ \frac{1}{2} \ln \left(\frac{u'^2}{p'_2} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\beta(\tau_\alpha) + \alpha(\tau_\alpha) e^{2\mathcal{Q}_\alpha^2}}{\delta(\tau_\alpha) + \gamma(\tau_\alpha) e^{2\mathcal{Q}_\alpha^2}} \right), \\ \frac{1}{2} \ln \left(\frac{v'^1}{\pi'_1} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\gamma(\tau_\alpha) + \delta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^3}}{\alpha(\tau_\alpha) + \beta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^3}} \right), \\ \frac{1}{2} \ln \left(\frac{v'^2}{\pi'_2} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\gamma(\tau_\alpha) + \delta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^4}}{\alpha(\tau_\alpha) + \beta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^4}} \right), \quad \alpha = 1, 2, \end{aligned} \quad (90)$$

which were obtained rewriting Eq. (87) in terms of (\vec{u}', \vec{p}') , $(\vec{v}', \vec{\pi}')$, and the gauge parameters.

In the gauge related variables, the gauge condition is

$$u'^1 = A, \quad u'^2 = B, \quad p'_1 = C. \quad (91)$$

By inserting Eq. (69) and using $\alpha\delta - \beta\gamma = 1$ the expressions of the gauge parameters are computed. Of course, they are the same as those found in the previous subsection, and given by Eq. (77). Notice again that $u^i(\tau_\alpha)$ and $p_i(\tau_\alpha)$ are parameters. The difference with the non-gauge-invariant case is that in the present case these parameters do not appear in the action, they appear in the boundary conditions only. Therefore the goal has been reached. The new variational principle is defined by the action (89) with the boundary conditions (90) where the gauge parameters are given by Eq. (77). In this variational principle the boundary conditions are compatible with the gauge conditions.

Now, as before, the gauge-fixed variational principle will be computed. To do this, the gauge conditions (91) and the constraint equations must be inserted into the action and into the boundary conditions. This gives a reduced action with, of course, a lower number of boundary conditions. By doing this, the variational principle is defined by the action

$$S_{\text{fixed}}[\vec{v}', \vec{\pi}'] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dv'^1}{d\tau} \pi'_1 + \frac{dv'^2}{d\tau} \pi'_2 \right] - \vec{v}' \cdot \vec{\pi}' \big|_{\tau_1}^{\tau_2} \quad (92)$$

under the boundary conditions

$$\begin{aligned} \frac{1}{2} \ln \left(\frac{v'^1}{\pi'_1} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\gamma(\tau_\alpha) + \delta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^3}}{\alpha(\tau_\alpha) + \beta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^3}} \right), \\ \frac{1}{2} \ln \left(\frac{v'^2}{\pi'_2} \right) (\tau_\alpha) &= \frac{1}{2} \ln \left(\frac{\gamma(\tau_\alpha) + \delta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^4}}{\alpha(\tau_\alpha) + \beta(\tau_\alpha) e^{2\mathcal{Q}_\alpha^4}} \right), \quad \alpha = 1, 2. \end{aligned} \quad (93)$$

The constraint $D' = 0$ was used to reduce the boundary term in the action. In this variational principle the phase space variables are $(\vec{v}', \vec{\pi}')$. Notice that there are no parameters in the action, rather, the parameters $u^i(\tau_\alpha)$, and $p_i(\tau_\alpha)$ are in the boundary conditions. The new thing here is that these parameters can be, using the first four equations in Eq. (87), the first four equations in Eq. (90), and the constraint $D' = 0$, put in terms of $A, B, C, \mathcal{Q}_\alpha^1, \mathcal{Q}_\alpha^2, v'^i(\tau_\alpha)$, and $\pi'_i(\tau_\alpha)$. Therefore the variational principle is well defined, its action has not a Hamiltonian and its dynamics sits both at the boundaries and in the boundary conditions. This might be interpreted as the canonical version of the holographic hypothesis [8].

III. WHAT IS A HAMILTONIAN CONSTRAINT?

Hamiltonian constraints are quadratic in the momenta. This fact implies that a boundary term arises when the gauge transformed action is computed. On the other hand, in Sec. II fully gauge-invariant actions were built in spite of the fact the systems have first class constraints quadratic in the momenta. Is there anything deep in fully gauge-invariant actions besides their aesthetic property? Is the gauge symmetry of generally covariant theories with first class constraints quadratic in the momenta of a different kind with respect to the one of Yang-Mills theories which have constraints linear in the momenta? In this section, new variational principles with first class constraints linear in the momenta will be built for the generally covariant systems with first class constraints quadratic in the momenta studied in Sec. II. These variational principles will be written in terms of the new phase space variables (Q^i, P_i) naturally induced by the boundary term. According to Eq. (5) with B replaced by $B + F$ the new variables are not unique and there is a freedom to select an appropriated combination of $B + F$ in such way that in the infinitesimal case the system of Eqs.(17) have a solution. Once, a solution of Eqs. (5) and (17) is found the fully gauge-invariant action is given by

$$S_{\text{inv}}[Q^i, P_i, \lambda^a] = \int_{\tau_1}^{\tau_2} d\tau \left(p_i \dot{q}^i - H_E - \frac{d}{d\tau}(B + F) \right), \quad (94)$$

subject to the boundary conditions

$$Q^i(\tau_\alpha) = Q_\alpha^i, \quad \alpha = 1, 2. \quad (95)$$

A. Parametrized relativistic free particle

The fully gauge-invariant action associated with the parametrized relativistic free particle is given by

$$S_{\text{inv}}[x^\mu, p_\mu, \lambda] = \int_{s_1}^{s_2} ds \left[\frac{dx^\mu}{ds} p_\mu - \lambda (p^\mu p_\mu + m^2) \right] - x^\mu p_\mu \Big|_{s_1}^{s_2}, \quad (96)$$

under the boundary conditions (34). The boundary term induces the canonical transformation (33) from the original set of variables (x^μ, p_μ) to the new phase space variables (Q^μ, P_μ) . By introducing the boundary term $-x^\mu p_\mu \Big|_{s_1}^{s_2}$ into the integral S_{inv} can be written in terms of the new phase space variables

$$S_{\text{inv}}[Q^\mu, P_\mu, \lambda] = \int_{s_1}^{s_2} ds \left[\frac{dQ^\mu}{ds} P_\mu - \lambda (\beta^2 Q^\mu Q_\mu + m^2 c^2) \right] \quad (97)$$

under the boundary conditions

$$Q^\mu(s_\alpha) = Q_\alpha^\mu. \quad (98)$$

Notice three things: (i) the action written in terms of Q^μ, P_μ has no boundary term, (ii) the first class constraint does not depend on the momenta P_μ , (iii) the action (97) transforms under gauge transformations in similar way to the action for self-dual gravity [2], whereas the transformation properties of the action (18) are equivalent to those of gravity in ADM variables.

B. Parametrized harmonic oscillator

The fully gauge-invariant action associated with the parametrized harmonic oscillator is given by

$$S_{\text{inv}}[x, t, p, p_t, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{dx}{d\tau} p + \frac{dt}{d\tau} p_t - \lambda \left(p_t + \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \right) \right] - \frac{1}{2} x p \Big|_{\tau_1}^{\tau_2} \quad (99)$$

under the boundary conditions (58). The boundary term induces the canonical transformation (57) from the initial canonical variables $(x, t; p, p_t)$ to the new canonical set $(\mathcal{X}, \mathcal{T}; \mathcal{P}, \mathcal{P}_\mathcal{T})$. By introducing the boundary term $-\frac{1}{2} x p \Big|_{\tau_1}^{\tau_2}$ into the integral the gauge-invariant action is written in terms of the new canonical variables

$$S_{\text{inv}}[\mathcal{X}, \mathcal{T}, \mathcal{P}, \mathcal{P}_\mathcal{T}, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{d\mathcal{X}}{d\tau} \mathcal{P} + \frac{d\mathcal{T}}{d\tau} \mathcal{P}_\mathcal{T} - \lambda \left(\mathcal{P}_\mathcal{T} + \frac{1}{2m} \mathcal{P} e^{-2\mathcal{X}} + \frac{1}{2} m \omega^2 \mathcal{P} e^{2\mathcal{X}} \right) \right], \quad (100)$$

under the boundary conditions

$$\mathcal{X}(\tau_\alpha) = \mathcal{X}_\alpha, \quad \mathcal{T}(\tau_\alpha) = \mathcal{T}_\alpha, \quad \alpha = 1, 2. \quad (101)$$

Notice two things: (i) Eq. (100) has no boundary term, (ii) the first class constraint in Eq. (100) is linear and homogeneous in the new momenta \mathcal{P} , $\mathcal{P}_\mathcal{T}$. These two facts are related. Due to the fact that the action (100) is fully gauge-invariant and has no boundary term, according to Ref. [1] the first class constraint has to be linear and homogeneous in the momenta as it really happens.

C. $SL(2, R)$ model

The fully gauge-invariant action which captures the $SL(2, R)$ gauge symmetry of this model is

$$S_{\text{inv}}[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{\vec{u}} \cdot \vec{p} + \dot{\vec{v}} \cdot \vec{\pi} - (NH_1 + MH_2 + \lambda D) \right] - \frac{1}{2}(\vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi}) \Big|_{\tau_1}^{\tau_2}, \quad (102)$$

with the boundary conditions (87). The boundary term induces a canonical transformation given by Eq. (86) from the initial set of canonical variables $(\vec{u}, \vec{v}; \vec{p}, \vec{\pi})$ to the new one $(\mathcal{Q}^i, \mathcal{P}_i)$. Then it is possible to rewrite the variational principle in terms of these new variables. This is done by introducing the boundary term into the integral in Eq. (102), and S_{inv} becomes

$$S_{\text{inv}}[\mathcal{Q}^i, \mathcal{P}_i, N, M, \lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[\frac{d\mathcal{Q}^i}{d\tau} \mathcal{P}_i - (NC_1 + MC_2 + \lambda C_3) \right], \quad (103)$$

with

$$\begin{aligned} C_1 &= \frac{1}{2} \left[\mathcal{P}_1 e^{-2\mathcal{Q}^1} + \mathcal{P}_2 e^{-2\mathcal{Q}^2} - \mathcal{P}_3 e^{2\mathcal{Q}^3} - \mathcal{P}_4 e^{2\mathcal{Q}^4} \right], \\ C_2 &= \frac{1}{2} \left[\mathcal{P}_3 e^{-2\mathcal{Q}^3} + \mathcal{P}_4 e^{-2\mathcal{Q}^4} - \mathcal{P}_1 e^{2\mathcal{Q}^1} - \mathcal{P}_2 e^{2\mathcal{Q}^2} \right], \\ C_3 &= \mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_3 - \mathcal{P}_4. \end{aligned} \quad (104)$$

under the boundary conditions

$$\mathcal{Q}^i(\tau_\alpha) = \mathcal{Q}_\alpha^i, \quad i = 1, 2, 3, 4, \quad \alpha = 1, 2. \quad (105)$$

Again, the new variational principle in terms of the phase space variables $(\mathcal{Q}^i, \mathcal{P}_i)$ features (i) no boundary term in the action (103), and (ii) the first class constraints are linear and homogeneous in the momenta \mathcal{P}_i .

It could be said in some sense that the parametrized harmonic oscillator and the $SL(2, R)$ model are of the same kind, both have first class constraints quadratic in the configuration and momentum variables. In both cases, the simplest boundary term needed to build S_{inv} is $-\frac{1}{2}q^i p_i$. The parametrized relativistic free particle, on the other hand, is quadratic in the momenta only. In that case, the boundary term needed to build S_{inv} is $-x^\mu p_\mu$ and S_{inv} is fully gauge-invariant on the constraint surface only. In all cases, when S_{inv} is written in terms of the phase space variables induced by the boundary term it happens that no boundary term is present anymore. In the cases where S_{inv} is fully gauge-invariant [parametrized harmonic oscillator and the $SL(2, R)$ model] the new constraints are *linear* and *homogeneous* in the new momentum variables whereas in the parametrized relativistic free particle where S_{inv} is gauge-invariant only on the constraint surface the new constraint is *not* homogeneous in the new momenta.

IV. CONCLUDING REMARKS

In this paper two main topics were touched on. The first one was the implementation of the ideas of Ref. [1] for gauge systems when *finite* gauge transformations are taken into account to make compatible both boundary and gauge conditions. In this case, also two different variational principles were analyzed. The first one features non-gauge-invariant actions whereas the other includes fully gauge-invariant ones. One of the main lessons learned from the finite but non-gauge-invariant approach is that the interplay between constraints quadratic and linear in the momenta can result in a contribution of the seconds to the boundary term in contrast to the infinitesimal approach.

The second contribution was to take advantage of fully gauge-invariant actions to rewrite such systems in terms of new phase space variables in terms of which the first class constraints are linear (and homogeneous in some cases) in the momenta. For a long time it has been considered that first class constraints quadratic in the momenta are the distinguishing mark of generally covariant theories, as general relativity or string theory [see, for instance, Ref. [4]]. Here it was shown that these kind of theories can be written in terms of first class constraints linear in the momenta if fully gauge-invariant actions are taken into account. Thus the gauge symmetry present in generally covariant theories with first class constraints quadratic in the momenta is apparently of the same kind as the gauge symmetry present in Yang-Mills theories if fully gauge-invariant actions are taken into account for the former: after all, both kinds of theories can be described with first class constraints linear in the momenta. It is important to recall that some steps in this direction have been done for Bosonic strings, at least infinitesimally [7]. In the case of general relativity, it has been shown in Ref. [2] that the action for gravity in terms of Ashtekar variables [12, 13, 14, 15, 16] is gauge-invariant up to a term proportional to the Hamiltonian constraint under the gauge symmetry generated by their first class constraints. This suggests that if one starts from the action for gravity in terms of Arnowitt-Deser-Misner (ADM) variables in its triad version [17] and apply Eq. (17), the solution for the generating function will be the one introduced in Ref. [18]. From the models studied here it is concluded that gauge invariance for the actions is the reason to have first class constraints linear in the momenta.

There are other aspects of this last topic which were not analyzed here but deserve to be studied. At a classical level, the introduction of the boundary term into the integral actions could have also been handled with the original set of canonical variables instead of making the canonical transformation as it was done here, but this other way of dealing with the boundary term in the fully gauge-invariant actions would have led to the introduction of second class constraints. On the other hand, it would be worth analyzing the quantum theory emerging from these systems with first class constraints linear in the momenta (and homogeneous in some cases) and compare with the standard quantization coming from their quadratic constraints. These issues are left for future work.

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